Dynamical systems: an integrable kernel for resonance effects

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Manuscript received on November 20, 2008 / accepted on January 12, 2009

ABSTRACT

A suitable sequence of canonical transformations reduces the system of differential equations describing the orbital motion to an integrable dynamic system. Through this dynamic system, the motion of an artificial satellite subject to geopotential perturbations and resonances between the frequencies of the mean orbital motion and the Earth rotational motion is analyzed. The behavior of the motion of the satellite is analyzed in the neighborhood of the 2:1 resonances. The phase space of the resulting system is studied considering that one resonant angle is fixed. Simulations are presented showing the time-behavior of the semi-major axis of artificial satellites.

Keywords: resonance, artificial satellites, celestial mechanics, dynamics systems.

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1 INTRODUCTION

The influence of resonances on the orbital and translational motion of artificial satellites has been extensively discussed in the literature under several aspects. Among the resonances that have been studied we quote:

(i) resonance of the rotation motion of a planet with the translational motion of the satellite (Lima Jr. 1998; Formiga 2005);

(ii) sun-synchronous resonance (Hughes 1980);

(iii) spin-orbit resonance (Beletskii 1975; Hamill and Blitzer 1974; Vilhena de Moraes et al. 1990);

(iv) resonances between the frequencies of the rotational motion of the satellite (Hamill and Blitzer 1974);

(v) resonance including solar radiation pressure perturbation (Ferraz-Mello 1979).

In this work, the type of resonance considered is the commensurability from resonance condition (Osório 1973):

\[ \frac{2}{l_1} = \frac{3}{l_2} = \cdots = \frac{m}{l_n} \]

where \( \mu \) is the gaussian constant; \( \lambda_{im} \) is the longitude of the semi major axis of symmetry for the spherical harmonic \( (\ell, m) \); \( J_{im} F^m_{\ell m}(L, G) \) are the Hansen's functions and \( F^m_{\ell m}(i) \) are the Kaula's inclination functions.

3 METHODOLOGY

Let us represent by \( n \) the mean motion of the satellite and by \( \alpha = q/m \) the commensurability from resonance condition

\[ qn - m \omega_0 = 0 \]

were \( q \) and \( m \) are integers.

The influence of the resonance on the orbital motion can be analyzed integrating a system of differential equations of the following type (Lima Jr. 1998; Formiga 2005):

\[
\frac{d(X, Y, Z, \Theta)}{dt} = \frac{\partial F}{\partial (X, Y, Z, \Theta)}, \\
\frac{d(x, y, z, \theta)}{dt} = -\frac{\partial F}{\partial (X, Y, Z, \Theta)}
\]

were the Hamiltonian \( F \) is

\[ F = \frac{\mu^2}{2L^2} + R_{\ell m p q} \]

and

\[ R_{\ell m p q} = \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=-\infty}^{\infty} \frac{\mu^2}{L^2} \left( \frac{\mu_0 q}{L} \right)^l J_{im} F^m_{\ell m}(L, G) \cos \psi_{\ell m p q}(\ell', g, h, \Theta) \]

and the argument is given by:

\[ \psi_{\ell m p q}(\ell', g, h, \Theta) = q \ell' + (\ell - 2p)g + m(h - \Theta - \lambda_{im}) + (\ell' - m)\frac{\pi}{2} \]

Also \( \mu = GM \approx 3.986009 \times 10^5 \text{km}^3/\text{s}^2 \) is the gaussian constant; \( \lambda_{im} \) is the longitude of the semi major axis of symmetry for the spherical harmonic \( (\ell, m) \); \( H_q^{(\ell+1),(-2p)}(\ell) \) are the Hansen's functions and \( F^m_{\ell m}(i) \) are the Kaula's inclination functions.
Explicit functions relating $B_{p + k + \text{imp} (\text{amu})}(X_1, C_1, C_2)$ and $F_{\text{imp}q}(X_1, C_1, C_2)$, with the keplerian elements can be found in Lima Jr. (1998).

4 NUMERICAL SIMULATIONS

Since Hansen’s coefficients have been used this theory can be applied for orbits of any eccentricity below one. In this chapter we present simulations considering a few initial conditions arbitrarily chosen.

4.1 Results for resonance 2:1

Let us consider the case $e = 0.01$, $i = 4^\circ$, $\phi^* = 0$ (critical angle) and the harmonics $J_2$ and $J_{22}$. Numerical values for the Harmonics coefficients $J_2$ and $J_{22}$ are given by JGM3.

The phase space for the dynamical system (6)–(7) is presented by Figure 1. This space is topologically equivalent to that of the simple pendulum. The behavior of the motion is here analysed in the neighborhood of the separatrix ($a$ about 26562.48 km).

Figure 2 represents the temporary change of the semi-major axis in the neighbourhood of the resonance 2:1. For several values considered for the semi-major axis, distinct behaviors for the time-changes can be observed. The more the satellites approaches the region that we defined as a resonant region, the more the variations increases. It is remarkable the oscillation in the region between $a = 26560.0$ km and $a = 26562.48$ km, that characterizes paths for which the effect of the resonance is maximum for the case where $e = 0.01$ and $i = 4^\circ$. For instance, for a small variation in the semi-axis, of about $10$ m ($a = 26562.49$ km) an abrupt decrease in $a$ is observed.

Figure 3 shows the temporary variation of the semi-major axis starting from initial conditions $a = a_o + 290$ km and the $a = a_o + 480$ km for $a_o = 26562.48$ km.

For the considered case, Figures 4 and 5 present, respectively, the behavior of the eccentricity ($0.0045 < \Delta e_{\text{max}} < 0.0065$) and inclination ($0.000004 < \Delta i_{\text{max}} < 0.000006$) in the neighborhood of the 2:1 resonance.
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Figure 2 – Variation of orbits in the time: $e = 0, \theta_1, i = 4^\circ$.

Figure 3 – Variation of orbits in the time: $e = 0, \theta_1, i = 4^\circ, \phi^* = 0^\circ$.

Figure 4 – Variation of eccentricity in the time: $e = 0, \theta_1, i = 4^\circ, \phi^* = 0^\circ$. 
The temporary variation of the critical angle is shown in Figure 6.

It is interesting to observe the behavior of the trajectories in the neighborhoods of the separatrix in the phase diagram given by Figure 6. Some paths were generated out of the resonant region. Still by, Figure 6, it can be observed the instant where the satellite is in the equilibrium position for $a = 26562.48$ km.

The stabilization of the orbit that appears after a period of 1420 days after a maximum oscillation (see Fig. 1), is related with a discontinuity produced by the resonance.

A possible region of stability for these conditions can also be observed by the behavior of the critical angle (Fig. 6). The satellite librates and circulate after suffering an abrupt discontinuity and starts to circulate in a region different from the initially considered.

5 CONCLUSIONS

In this work, based on Lima Júnior’s theory (Lima Jr. 1998) a integrable kernel was found. The theory is valid for any type resonance $p/q$, and was here applied in a few cases.

The motion near the region of the exact resonance, is extremely sensitive to the small alterations considered. This can be indicative that these regions are chaotic.

This work provides a good approach for long period orbital evolution studies of satellites orbiting in regions where the influence of the resonance is more pronounced.
ACKNOWLEDGMENTS
This work was supported by CNPq and by Brazilian Institute for Space Research.

REFERENCES


